Pub-83/40-T
SLAC-PUB-3011
November 1982
(T/E)

ON THE ELIMINATION OF SCALE AMBIGUITIES IN PERTURBATIVE QUANTUM CHROMODYNAMICS*

Stanley J. Brodsky
Institute for Advanced Study
Princeton University, Princeton, New Jersey 08540
and
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305†

G. Peter Lepage
Institute for Advance Study
Princeton University, Princeton, New Jersey 08540
and
Laboratory of Nuclear Studies
Cornell University, Ithaca, New York 14853†

Paul B. Mackenzie
Fermilab, Batavia, Illinois 60510

(Submitted to Physical Review D)

*Work supported in part by the Department of Energy, contract number DE-AC03-76SF00515 and

†Current address.

ABSTRACT

We present a new method for resolving the scheme-scale ambiguity that has plagued perturbative analyses in quantum chromodynamics (QCD) and other gauge theories. For Abelian theories the method reduces to the standard criterion that only vacuum polarization insertions contribute to the effective coupling constant. Given a scheme, our procedure automatically determines the coupling-constant scale appropriate to a particular process. This leads to a new criterion for the convergence of perturbative expansions in QCD. We examine a number of well known reactions in QCD, and find that perturbation theory converges well for all processes other than the gluonic width of the Υ. Our analysis calls into question recent determinations of the QCD coupling constant based upon Υ decay.

1. INTRODUCTION

A major ambiguity in the interpretation of perturbative expansions in quantum chromodynamics (QCD) is in the choice of an expansion parameter. In general QCD predictions for some measurable quantity ρ have the form

$$\rho = C_0\alpha_s(Q)\left\{1 + C_1(Q)\,\frac{\alpha_s(Q)}{\pi} + C_2(Q)\,\frac{\alpha_s^2(Q)}{\pi^2} + \cdots\right\} \qquad (1)$$

The coefficients $C_i(Q)$ depend upon both the exact definition of the running coupling constant $\alpha_s(Q)$ (i.e., the 'scheme'), and upon the *choice of scale Q. When working to all orders in $\alpha_s(Q)$* the choice of scheme and scale is irrelevant; the coefficients $C_i(Q)$ are defined so that ρ is the same for all choices. However, this freedom can be a serious source of confusion in finite order analyses. Indeed when working to first order, one can set $C_1(Q)$ to any value simply by redefining α_s or by changing Q. This coefficient seems meaningless here. In particular it seems to give no indication of the convergence of the expansion. This question is of critical importance in testing QCD since α_s is rather large (~.1-.3) at current energies. It is quite likely that perturbation theory will fail completely for some processes. Such processes must be identified.

The potential difficulties are well illustrated in low energy quantum electrodynamics (QED), where for example the electron anomaly has a very convergent expansion,

$$a_e = \frac{g_e - 2}{2} = \frac{\alpha}{2\pi}\left[1 - 0.657\,\frac{\alpha}{\pi} + 2.352\,\frac{\alpha^2}{\pi^2}\,\cdots\right] \qquad (2)$$

while the expansion for orthopositronium decay is much less convergent:

$$\Gamma_{0\text{-}Ps} = \Gamma_0\left\{1 - 10.3\,\frac{\alpha}{\pi} + \cdots\right\} \quad . \qquad (3)$$

The difference in convergence rate here is not an artifact due to a bad choice of scheme or scale; the first order coefficients in these expansions should not be absorbed into a redefinition of α since the running coupling constant for QED doesn't run at these energies (i.e., below the e^+e^- threshold).

While numerous schemes have been studied (MS,[1] $\overline{\mathrm{MS}}$,[2] MOM,[3] $\cdots$), little has been done to resolve the scale ambiguity. In this paper we introduce an automatic procedure for determining the coupling-constant scale appropriate to a particular process.[4] Given a scheme, this results in a new criterion for the convergence of perturbative expansions in QCD by unambiguously fixing the expansion coefficient $C_1(Q)$ in Eq. (1) for a given process; *perturbation theory cannot be trusted when $C_1(Q)$ $\alpha_s(Q)/\pi \gtrsim 1$.* Furthermore, the coupling-constant scale can be determined without computing all higher order corrections. <u>Thus leading order analyses in QCD can be meaningfully compared with experiments</u>.

In Section 2, we outline our basic approach as applied to QED (i.e., Abelian theories). We define the running coupling constant $\alpha(Q)$ for QED to include all contributions due to vacuum polarization insertions in the photon propagator. This is the only natural choice since the variation of the effective coupling in QED is due to vacuum polarization alone. The coupling-constant scale Q^* best suited to a particular process in a given order can be determined simply by computing the vacuum-polarization insertions in the diagrams of that order. Expansion (1) is then replaced by

$$\rho = C_0\alpha(Q_0^*)\left\{1 + C_1^*\,\frac{\alpha(Q_1^*)}{\pi} + C_2^*\,\frac{\alpha^2(Q_2^*)}{\pi^2} + \cdots\right\} \tag{4}$$

where all photon self-energy corrections are absorbed into the effective coupling constants by an appropriate (and unique) choice of scales Q_0^*, Q_1^*, $\cdots$. Since all dependence upon the number of light-fermion flavors (n_f) usually enters through the photon self-energy in low orders, both the coupling-constant scales Q_i^* and the low order coefficients C_i^* are independent of n_f. (Light-by-light scattering graphs lead to n_f dependence in higher orders.) The light-fermion loop corrections serve mainly to renormalize $\alpha(Q)$, as expected. Note also, that in a general process, the scales Q_0^*, Q_1^* can depend on the ratio of invariants, e.g., center-of-mass angles.

In QCD (i.e., non-Abelian theories), it again is natural to absorb all vacuum polarization corrections into $\alpha_s(Q)$. In particular, all vacuum polarization due to light fermions should be absorbed, leaving an expansion

$$\rho = C_0 \alpha_s(Q^*) \left[1 + C_1^* \frac{\alpha_s(Q^*)}{\pi} + \cdots\right] \tag{5}$$

where C_1^* and Q^* are defined to be n_f independent. (The calculation of C_1^* and Q^* is unambiguous since the dependence of α_s on n_f is determined to this order by $\beta_0 = 11 - 2/3\, n_f$.) Although the scale Q^* is now automatically fixed, the expansion (5) still depends upon the definition of $\alpha_s(Q)$ – i.e., upon the renormalization 'scheme'.[5] One can easily create schemes in which C_1^* is arbitrarily large, and, unlike QED, QCD has no scheme which is obviously superior. This scheme ambiguity can in fact be eliminated to a large extent by adopting some physical process as a theoretical standard for defining $\alpha_s(Q)$.[6] For example, the ratio of $e^+e^- \to \mu^+\mu^-$ might be defined to be exactly ($s = Q^2$)

$$R_{e^+e^-}(Q^2) \equiv 3 \sum_{\text{flavors}} e_q^2 \left[1 + \frac{\alpha_R(Q)}{\pi}\right] . \qquad (6)$$

Expansions for other procedures would then be expressed in terms of $\alpha_R(Q^*)$, with scale Q^* chosen such that C_1^* and Q^* are independent of n_f as in Eq. (5). As it happens, expressions derived in this R-scheme are almost identical to those obtained for the MS or $\overline{\text{MS}}$ schemes (MS and $\overline{\text{MS}}$ give the same expansions when used with our procedure). We will adopt the $\overline{\text{MS}}$ scheme as our standard in this paper, since it is the more familiar.

There are a large number of physical processes which could be used as the standard for defining α_s with similar qualitative results. A bad choice for the standard process can be detected immediately upon application. This is because the differences between first order coefficients C_1^* for various processes are independent of the scheme; therefore, for a bad choice of standard process most coefficients C_1^* will be large and have the same sign. This, in fact, does not seem to be the case for the R - MS - $\overline{\text{MS}}$ scheme, since for a large number of processes the coefficients C_1^* obtained by using the automatic scale fixing procedure are indeed small.

The plan of this paper is as follows. In the next section we review the procedure in which the scale of the running coupling coupling constant is set in Abelian gauge theory. These ideas are then developed for QCD in Section 3. We limit our discussion to lowest and first-order corrections, and focus upon processes that do not involve a gluon-gluon coupling in leading order. This is sufficient for most phenomenologically relevant processes in QCD, and we illustrate our procedure for a number of well known reactions. Most significantly, we find that the gluonic width of the Υ has a very unreliable perturbative expansion. *We also find that first order corrections are numerically* small ($\lesssim$10-20 percent) for all of the other processes considered, when the correct coupling-constant scale Q^* is employed; the lowest order calculations, together with the quark vacuum polarization corrections which set Q^*, are quite adequate in these cases for a quantitative *comparison of theory and experiment.*

Finally, we summarize our results in Section 4, contrasting our approach with other attempts at resolving the "scheme-scale ambiguity." We also briefly explore the possibility of generalizing our method so that it may be applied to all processes in QCD.

2. QED (ABELIAN GAUGE THEORIES)

The only true ultraviolet divergences in QED are associated with vacuum polarization, because divergences in the vertex and fermion self-energy corrections cancel by the Ward identity (or are absent in Landau gauge). Thus it is only vacuum polarization corrections that renormalize the coupling. Since these corrections vanish like Q^2/m_e^2 as $Q^2 \to 0$, QED becomes a fixed point theory at very low energies[7]:

$$\alpha(Q) \to \alpha = 1/137.036 \cdots \quad \text{as } Q \to 0 \quad . \tag{7}$$

Equation (7) serves as an initial condition for the renormalization group equations, which then uniquely determine $\alpha(Q)$ for all Q. In effect, we are absorbing the entire vacuum polarization correction into $\alpha(Q)$ – i.e., ($Q^2 = -q^2$)

$$\alpha(Q)\ \frac{-g^{\mu\nu} + q^{\mu}q^{\nu}/q^2}{q^2 + i\epsilon} = \alpha_0(\Lambda)\ d^{\mu\nu}(q,\Lambda) \tag{8}$$

where $\alpha_0(\Lambda)$ is the bare coupling and $d^{\mu\nu}$ the unrenormalized photon propagator.

Given this definition, we need only determine the appropriate scale (or scales) Q for a given process. The most naive procedure is simply to use the full propagator (Eq. (8)) for each photon in any given diagram.[8] For example, we can replace α by $\alpha(Q)$ for (with $Q^2 = -q^2$) before integrating over q in the leading diagram for the muon anomaly (Fig. 1a).[9] All vacuum polarization insertions are automatically included. Unfortunately, the loop integration is then quite cumbersome. However, by the mean value theorem there must be some scale $Q^* \sim m_\mu$ for which the exact result is

$$a_\mu{}^{VP} = \frac{\alpha(Q^*)}{2\pi} \tag{9a}$$

where from the definition, Eq. (8)

$$\alpha(Q) = \frac{\alpha}{1 - \frac{\alpha}{\pi}\left(\frac{2}{3}\,\ell n\,\frac{Q}{m_e} - \frac{5}{3}\right) - \left(\frac{\alpha}{\pi}\right)^2\left(\frac{1}{2}\,\ell n\,\frac{Q}{m_e} + \zeta(3) - \frac{25}{24}\right) - \cdots} \tag{9b}$$

(For simplicity we are neglecting muon loops and terms of order m_e/Q or less in $\alpha(Q)$.) The scale Q^* can then be determined order by order in perturbation theory by expanding (9) in powers of α and adjusting the coefficients to agree with results obtained order by order from vacuum polarization insertions in the basic diagram. For example, the lowest order electron loop (Fig. 1b) contributes

$$A_{VP}\,\frac{\alpha}{\pi}\,a_\mu{}^0 = \left(\frac{2}{3}\,\ell n\,\frac{m_\mu}{m_e} - \frac{25}{18}\right)\frac{\alpha}{\pi}\,a_\mu{}^0$$

which from Eq. (9a) must equal

$$\left[\frac{2}{3}\ \ln\frac{Q^*}{m_e}-\frac{5}{3}\right]\frac{\alpha}{\pi}\ a_\mu{}^0 \quad .$$

Thus we have $Q^* = m_\mu e^{5/12}$ in leading order. With this procedure, the muon anomaly has the same expansion to first order as the electron anomaly (Eq. (2)) but with a different expansion parameter — i.e., to this order we are replacing

$$a_\mu = \frac{\alpha}{2\pi}\left[1+\frac{\alpha}{\pi}\ (A_{VP} + C_1) + \cdots\right]$$

by

$$a_\mu = \frac{\alpha(Q^*)}{2\pi}\left[1+\frac{\alpha(Q^*)}{\pi}\ C_1 + \cdots\right] \tag{10a}$$

where

$$\alpha(Q^*) \simeq \frac{\alpha}{1-\dfrac{\alpha}{\pi}\ A_{VP}} \tag{10b}$$

and

$$C_1 = \frac{197}{72}+\frac{\pi^2}{6}-\pi^2\ln 2+\frac{3}{2}\ \zeta(3) \cong -0.657 \quad .$$

Intuitively this is reasonable since if a single insertion gives $\alpha/\pi\ A_{VP}$, a double insertion will give roughly $(\alpha/\pi\ A_{VP})^2$, and so on. Thus the electrons modify only the charge and not the physical expansion of a_μ in this order. Of course this is no longer the case in higher orders, when 'light-by-light' diagrams (Fig. 1c) and others like them appear.

The physical scale Q^* is refined by higher order corrections — $Q^* = m_\mu e^{5/12}\,(1 + 1.14\,\alpha/\pi + \cdots)$ — but its expansion is obviously far more convergent than the original expansion for a_μ. Also this expansion is unique. For example, including the $C_1\,\alpha/\pi$ in $\alpha(Q^*)$ (Eq. (10b)) would wreak havoc with the next-to-leading logarithms of m_μ/m_e in higher orders; there is no reason to expect that the $C_1\,\alpha/\pi$ is part of an approximately geometric series of contributions, while the vacuum polarization corrections must be geometric (for renormalizability). Finally, each order in perturbation theory will usually have its own scale (determined as above); there is no reason for all running couplings to have the same scale.

3. QCD (NON-ABELIAN GAUGE THEORIES)

A natural definition for the running coupling has proven far more elusive in QCD than in QED. There is no boundary condition for $\alpha_s(Q)$ analogous to Eq. (7). A perverse definition — e.g., $\alpha_P(Q) = \alpha_{\overline{MS}}(Q) + 10^6\,\alpha_{\overline{MS}}(Q)$ — would lead to absurd results. To avoid or at least minimize this possibility we can define $\alpha_s(Q)$ directly in terms of a specific physical process, as in Eq. (6). This is equivalent to prescribing a renormalization scheme. Here, however, we will simply adopt the $\overline{MS}$ scheme, since it happens to be practically equivalent to choosing $R_{e^+e^-}$ to define α_s.

Our procedure for fixing the scale is then straightforward, at least for processes which do not have gluon-gluon interactions in lowest order. To first order, such a process has an expansion

$$\rho = C_0\, \alpha_{\overline{MS}}(Q) \left[1 + \frac{\alpha_{\overline{MS}}(Q)}{\pi} (A_{VP} n_f + B)\right]$$

where the n_f term is all due to quark vacuum polarization. As in QED, the sole function of these light-quark insertions is to renormalize the coupling. Given a reasonable scheme, all such terms should be completely absorbed into the leading order coupling by redefining the scale:

$$\alpha_{\overline{MS}}(Q) \rightarrow \alpha_{\overline{MS}}(Q^*) = \alpha_{\overline{MS}}(Q) \left[1 + \frac{\beta_0}{2\pi} \alpha_{\overline{MS}}(Q)\, \ln \frac{Q^*}{Q} + \cdots\right]^{-1} .$$

Furthermore, the new scale Q^* must be n_f-independent if it is to retain any physical significance in relation to the momenta circulating in the leading order diagrams. Thus we replace

$$\rho = C_0\, \alpha_{\overline{MS}}(Q) \left[1 + \frac{\alpha_{\overline{MS}}(Q)}{\pi} \left\{-\frac{3}{2} \beta_0 A_{VP} + \frac{33}{2} A_{VP} + B\right\} + \cdots\right]$$

by

$$\rho = C_0\, \alpha_{\overline{MS}}(Q^*) \left[1 + \frac{\alpha_{\overline{MS}}(Q^*)}{\pi} C_1^* + \cdots\right] \tag{11a}$$

where

$$Q^* = Q \exp(3\, A_{VP})$$

$$C_1^* = 33/2\, A_{VP} + B \quad . \tag{11b}$$

The term $33\, A_{VP}/2$ in C_1^* serves to remove that part of the constant B which renormalizes the leading order coupling. The ratio of these gluonic corrections to the light-quark corrections is fixed by $\beta_0 = 11 - 2/3\, n_f$.

Several features of this procedure are worth noting:

— Two schemes that differ only by an n_f-independent rescaling give identical expansions in $\alpha_s(Q^*)$. Thus the differences between MS and $\overline{\rm MS}$, for example, are irrelevant in this approach. Furthermore, $\alpha_R(Q)$ could be replaced by $\alpha_R(Q/2)$ or $\alpha_R(10^{20}Q)$ in definition (6) with *no* effect on the final results for any process *ρ* expressed in terms of $\alpha_R(Q^*)$.

— If the $\overline{\rm MS}$ scheme is replaced by another for which

$$\begin{aligned}\alpha_s(Q) &= \alpha_{\overline{\rm MS}}(Q)\left[1 + \frac{\alpha_{\overline{\rm MS}}}{\pi}(D\,B_0 + E) + \cdots\right] \\ &= \alpha_{\overline{\rm MS}}(Q\,e^{-2D})\left[1 + \frac{\alpha_{\overline{\rm MS}}}{\pi}E + \cdots\right]\end{aligned} \tag{12}$$

where D and E are n_f-independent, then the first order coefficients for all processes are shifted by $-E$: $C_1^* \to C_1^* - E$. Differences between first order coefficients are scheme independent. Thus, for a poorly chosen scheme, the coefficients for most processes will be large and have the same sign. On the other hand, if several *processes have convergent expansions (i.e., C_1^* small) in some* scheme, then this will still be true in the physical scheme defined in terms of any one of these processes (see Eq. (6)).

— The leading order scale is determined solely by A_{VP}, which comes from quark vacuum-polarization insertions. This is usually all that need be computed to make a meaningful leading order *prediction, as we show below*.

— Equation (11a) is a particularly convenient way to present perturbative results since all flavor dependence is implicit in the definition of $\alpha_{\overline{\rm MS}}$.

The automatic scale-fixing procedure determines a natural expansion parameter $\alpha_{\overline{MS}}(Q^*)$ for the majority of interesting processes in QCD. However, reactions with gluon-gluon couplings in leading order are more difficult to analyze because quark loops appear in the first-order radiative corrections to the gluon-gluon vertex as well as in propagator insertions. It seems difficult if not impossible to separate the divergent part of the vertex, which renormalizes α_s, from the finite process-dependent part in any unique and general fashion. Consequently, our procedure for determining Q^* is inapplicable; not all of the n_f-dependence should be absorbed into $\alpha_s(Q^*)$. Since any process involves gluon-gluon vertices in first order and beyond, we presently can determine Q^* only to lowest order in α_s/π.

To illustrate our scale-fixing procedure and to explore its implications, we examine briefly a number of well known predictions of QCD:

<u>$e^+e^- \to$ hadrons</u> — The ratio of the total cross section into hadrons to the cross section for $e^+e^- \to \mu^+\mu^-$ is ($s = Q^2$)[10]

$$R_{e^+e^-}(Q^2) = 3\sum_q e_q^2 \left\{1 + \frac{\alpha_{\overline{MS}}(Q)}{\pi} + \frac{\alpha_{\overline{MS}}^2}{\pi^2}\,(1.98 - 0.115\, n_f) + \cdots\right\} \tag{13a}$$

$$\to 3\sum_q e_q^2 \left\{1 + \frac{\alpha_{\overline{MS}}(Q^*)}{\pi} + \frac{\alpha_{\overline{MS}}^2(Q^*)}{\pi^2}\,0.08 + \cdots\right\} \tag{13b}$$

where from Eq. (11), $Q^* = 0.71Q$. Notice that $\alpha_R(Q)$ (Eq. (6)) differs from $\alpha_{\overline{MS}}(Q^*)$ by only $0.08\,\alpha_{\overline{MS}}/\pi$, so that effectively $\alpha_R(Q)$ and $\alpha_{\overline{MS}}(.71\,Q)$ are interchangeable (for any value of n_f).

Deep Inelastic Scattering – The moments of the non-singlet structure function $F_2(x,Q^2)$ obey the evolution equation[11]

$$Q^2 \frac{d}{dQ^2} \ln M_n(Q^2) = -\frac{\gamma_n^{(0)}}{8\pi} \alpha_{\overline{MS}}(Q) \left\{1 + \frac{\alpha_{\overline{MS}}}{4\pi} \frac{2B_0 B_n + \gamma_n^{(1)}}{\gamma_n^{(0)}} + \cdots\right\} \tag{14a}$$

$$\rightarrow -\frac{\gamma_n^{(0)}}{8\pi} \alpha_{\overline{MS}}(Q_n^*) \left\{1 - \frac{\alpha_{\overline{MS}}(Q_n^*)}{\pi} C_n + \cdots\right\} \tag{14b}$$

where, for example,

$$Q_2^* = 0.48\ Q \qquad C_2 = 0.27$$

$$Q_{10}^* = 0.21\ Q \qquad C_{10} = 1.1\ .$$

For n very large, the effective scale here becomes $Q_n^* \sim Q/\sqrt{n}$ which is exactly what was found in Ref. 12 by a detailed study of the kinematics of deep inelastic scattering.

η_c Decay – The ratio of the η_c width into hadrons to that into $\gamma\gamma$ is $(n_f = 3)$[13]

$$\frac{\Gamma(\eta_c \rightarrow \text{hadrons})}{\Gamma(\eta_c \rightarrow \gamma\gamma)} = \frac{2}{9e_c^4} \frac{\alpha_{\overline{MS}}^2(M_{\eta_c})}{\alpha_{QED}^2} \left\{1 + \frac{\alpha_{\overline{MS}}}{\pi} \left[17.13 - \frac{8}{9} n_f\right] + \cdots\right\} \tag{15a}$$

$$\rightarrow \frac{2}{9e_c^4} \frac{\alpha_{\overline{MS}}^2(M^*)}{\alpha_{QED}^2} \left\{1 + \frac{\alpha_{\overline{MS}}(M^*)}{\pi} 2.46 + \cdots\right\} \tag{15b}$$

where $M^* = 0.26\ M_{\eta_c}$.

Υ Decay – The ratio of the hadronic to the leptonic widths of the Υ is $(n_f = 4)$[14]

$$\frac{\Gamma(\Upsilon \to \text{hadrons})}{\Gamma(\Upsilon \to \mu^+\mu^-)} = \frac{10(\pi^2 - 9)}{81\pi\, e_b^2} \frac{\alpha_{\overline{MS}}^3(M_\Upsilon)}{\alpha_{QED}^2} \left\{1 + \frac{\alpha_{\overline{MS}}}{\pi} (2.770(7)\beta_0 - 14.0(5)) + \cdots\right\} \tag{16a}$$

$$\to \frac{10(\pi^2 - 9)}{81\pi e_b^2} \frac{\alpha_{\overline{MS}}^3(M^*)}{\alpha_{QED}^2} \left\{1 - \frac{\alpha_{\overline{MS}}(M^*)}{\pi} 14.0(5) + \cdots\right\} \tag{16b}$$

where $M^* = 0.157\ M_\Upsilon$. Thus the decay rate into gluons has a large negative correction with this physical definition of the coupling, just as do the rates for $\Upsilon \to \gamma\gamma\gamma$ and for orthopositronium decay into three photons, both of which are scheme and scale independent to this order. Such a correction implies large, positive terms in higher orders, and in fact these are necessary if we are to fit the data. Further study is clearly necessary before Υ decay can be used as a reliable measure of α_s. We do note, however, that the large corrections cancel almost completely in the branching ratio for producing a direct photon plus hadrons[15]:

$$\frac{\Gamma(\Upsilon \to \gamma_D + \text{hadrons})}{\Gamma(\Upsilon \to \text{hadrons})} = \frac{36\ e_b^2}{5} \frac{\alpha_{QED}}{\alpha_{\overline{MS}}(M^*)} \left\{1 + \frac{\alpha_{\overline{MS}}(M^*)}{\pi} 2.2(6) + \cdots\right\} \tag{17}$$

where again $M^* = 0.157\ M_\Upsilon$. This cancellation occurs because the leading order amplitudes for $\Upsilon \to ggg$ and $\Upsilon \to \gamma gg$ are identical in structure. Thus the branching ratio for direct photons could be used to determine $\alpha_{\overline{MS}}$.

Exclusive Processes — Exclusive processes involving large transverse momentum are given by the convolution of distribution amplitudes $\phi(x,Q)$, representing the wavefunctions of each initial and final state hadron, with (collinear irreducible) hard scattering amplitudes $T_H(x_i,Q)$ in which each hadron is replaced by collinear on-shell quarks (or gluons).[16] The procedure given above allows us trivially to include the vacuum polarization corrections to the (skeleton) tree graphs contributing to T_H, and thus set the coupling-constant scale for the leading order results. For example, the hard scattering amplitude required for the form factor of helicity zero mesons is (Fig. 2)

$$T_H(x,y,Q) = \frac{64\pi}{3Q^2} \frac{\alpha_{\overline{MS}}\,(e^{-5/6}\sqrt{(1-x)(1-y)}\;Q)}{(1-x)(1-y)} \tag{18}$$

since the gluon's momentum transfer is $-(1-x)(1-y)Q^2$ (Fig. 3). If we estimate $\langle x\rangle \sim \langle y\rangle \sim 1/2$, then the correct expansion parameter for T_H is $\sim \alpha_{\overline{MS}}(Q/4.6)$ in agreement with the detailed analysis in Ref. 17.

<u>$Q\bar{Q}$ Potential</u> – The interaction potential between two infinitely massive quarks is[18]

$$V(Q^2) = -\frac{C_F\;4\pi\;\alpha_{\overline{MS}}(Q)}{Q^2}\left\{1 + \frac{\alpha_{\overline{MS}}}{\pi}\left[\frac{5}{12}\,\beta_0 - 2\right] + \cdots\right\} \tag{19a}$$

$$\rightarrow -\frac{C_F\;4\pi\;\alpha_{\overline{MS}}(Q^*)}{Q^2}\left\{1 - \frac{\alpha_{\overline{MS}}(Q^*)}{\pi}\,2 + \cdots\right\} \tag{19b}$$

where $Q^* = e^{-5/6}\,Q \cong 0.43\,Q$. This result shows that the effective scale of the $\overline{MS}$ scheme is about half of the true momentum transfer occuring in the interaction potential. In parallel to QED, the effective potential $V(Q^2)$ gives a particularly intuitive scheme for defining the QCD coupling constant

$$V(Q^2) \equiv -\frac{4\pi\;C_F\;\alpha_V(Q)}{Q^2} \tag{20}$$

with $\alpha_V(Q) = \alpha_{\overline{MS}}(e^{-5/6}Q)\left\{1 - 2\alpha_{\overline{MS}}/\pi \cdots\right\}$. The perturbative QCD prediction can be tested empirically — without scheme or scale ambiguities — if the predictions for two processes such as (6) and (19) are consistent with experiment.

<u>$\alpha_{MOM}(Q)$</u> — The standard MOM definition of α_s is (Landau gauge)[3]

$$\alpha_{MOM}(Q) = \alpha_{\overline{MS}}(Q)\left\{1 + \frac{\alpha_{\overline{MS}}}{\pi}(1.28\,\beta_0 - 7.47) + \cdots\right\} \tag{21a}$$

$$\rightarrow \alpha_{\overline{MS}}(Q^*)\left\{1 - \frac{\alpha_{\overline{MS}}(Q^*)}{\pi}\,7.47 + \cdots\right\} \tag{21b}$$

where $Q^* = 0.077\,Q$. Although this is not a physical process, we include this result because MOM is a widely used scheme. Clearly the MOM scheme is incompatable with our method of fixing Q^*; all first order coefficients would be increased by 7.47 if MOM replaced $\overline{MS}$. This is not unexpected since α_{MOM} is defined in terms of the tri-gluon interaction and such processes are specifically excluded from our analysis. Indeed the $\overline{MOM}$ scheme based upon the quark-gluon vertex is a perfectly acceptable alternative to $\overline{MS}$[19]:

$$\alpha_{\overline{MOM}}(Q) = \alpha_{\overline{MS}}(Q^*)\left\{1 - \frac{\alpha_{\overline{MS}}(Q^*)}{\pi}\,0.4 + \cdots\right\} \tag{22}$$

where $Q^* = 0.43\,Q$ and Landau gauge is assumed. It is only accidental that $\alpha_{MOM}(Q)$ and $\alpha_{\overline{MOM}}(Q)$ are nearly identical for $n_f = 4$. This is not the case for $n_f \neq 4$ ($n_f = 0 \Rightarrow \alpha_{MOM}(Q) = \alpha_{\overline{MOM}}(Q)(1 + 2.4\,\alpha_{\overline{MOM}}/\pi)$) and from our perspective the $\overline{MOM}$ definition is preferable.

4. CONCLUSIONS

A striking feature of each of the perturbative QCD predictions, is that — except for Υ decay — the first order correction in $\alpha_{\overline{MS}}$ is only 10 to 20 percent of the leading term at typical Q^2 after the scale has been fixed. (This is despite the fact that the coefficient $A_{VP}n_f + B$ is replaced by $33/2\ A_{VP} + B$, as in Eq. (11).) Perturbation theory seems to work rather well — the leading term in $\alpha_{\overline{MS}}(Q^*)$ for these processes is by itself quite accurate. The main effect of the higher order corrections is in setting the correct scale Q^*, and for this only the fermionic vacuum polarization corrections are needed. In effect the automatic scale fixing procedure uses the fermionic loops to probe the momentum flowing in the leading order diagrams. The remainder of the higher order corrections, i.e., the $(33/2\ A_{VP} + B)\alpha_s/\pi$, must of course be computed to obtain predictions with precision better than 10 to 20 percent.

For Υ decay into three gluons (Eq. (16)), the higher order corrections are quite large, calling into question the possibility of a perturbative analysis of this reaction. The fact that the higher order corrections for the corresponding decay of orthopositronium in QED are large (see Eq. (3)) indicates that this effect is not due to ambiguities in the renormalization scale.

The automatic scale-fixing procedure given in this paper is applicable for any choice of renormalization scheme. However, once the scale-fixing procedure is used, we can readily normalize and thus define $\alpha_s(Q)$ by using a convenient physical process such as $R_{e^+e^-}(Q^2)$ (Eq. (6)) or the effective potential $V(Q^2)$ between heavy quarks (Eq. 19)). Since the first order corrections are small in $\overline{MS}$ scheme, any one of the physical processes considered could have been used to define α_s, with essentially the same results. The exception is again Υ decay. Rewriting the other expansions in terms of α_Υ, defined such that

$$\frac{\Gamma(\Upsilon \to \text{hadrons})}{\Gamma(\Upsilon \to \mu^+\mu^-)} = \frac{10(\pi^2 - 9)}{81\ \pi\ e_b^2}\ \frac{\alpha_\Upsilon^3(M_\Upsilon)}{\alpha_{QED}^2} \tag{23}$$

is exact, results in first order corrections ranging from $+3\ \alpha_\Upsilon/\pi$ to $+7\ \alpha_\Upsilon/\pi$, depending upon the process. This seems not to be a very good scheme. The standard MOM scheme appears to be even worse and so is incompatible with our technique.

In the past, two viewpoints have prevailed concerning the resolution of the scheme-scale ambiguity. One was simply to adopt some definition of the coupling (MS, $\overline{MS}$, MOM, ...) and then attempt to guess the appropriate scale for the process under study (e.g., $Q^* = M_\Upsilon/3$ for Υ decay since there are three gluon jets in leading order). Our procedure removes any guesswork by automatically determining the scale. It is an essential complement to any analysis of scheme dependence. Furthermore, we now can easily introduce physical schemes for defining α_s (e.g., Eq. (6)) which are both gauge independent (unlike MOM) and regulator independent (unlike MS, $\overline{MS}$).

The second viewpoint holds that for want of better guidance we should adopt some ad hoc principle such as maximal convergence,[20] where Q^* is chosen so that $C_i(Q^*) = 0$ for $i \geq 1$ in Eq. (1), or minimal sensitivity,[21] where Q^* is chosen to minimize the variation of ρ with Q^* (due to omission of higher order terms in Eq. (1)). Unlike our procedure, these methods give no indication of the convergence of perturbation theory; $C_1(Q^*)$ is by definition small and process independent for both of the methods mentioned above. Such methods will usually be completely wrong when applied to processes, like Υ decay, for which the higher order corrections are very large; worse, they give no warning of such situations.

Our scale-fixing procedure is obviously far from complete. The most pressing problem is to find a suitable method for analyzing processes with gluon-gluon couplings in lowest order. An interim procedure might be to absorb all fermion loop corrections – i.e., vacuum-polarization, quark loops coupled to three gluons, etc. – into the coupling constant, while using some definition of α_s related to the tri-gluon interaction (e.g., α_{MOM}). However, something better should be found. When it has been, the extension of our analysis to higher orders will be straightforward (as is already true in QED).

Acknowledgements

We thank W. Celmaster and P. Stevenson for their comments and criticisms of an earlier draft of this paper. We would also like to thank D. J. Broadhurst, M. Moshe, Y. J. Ng, and J. Stirling for helpful conversations.

References

1. G. 't Hooft, Nucl. Phys. B61, 455 (1973).
2. W. Bardeen, A. Buras, D. Duke and T. Muta, Phys. Rev. D 18, 3998 (1978).
3. W. Celmaster and R. J. Gonsalves, Phys. Rev. D 20, 1420 (1979).

4. Brief reports on this method are given in G. P. Lepage, S. J. Brodsky, T. Huang, P. B. Mackenzie, to be published in the Proceedings of the Banff Summer Institute on Particles Physics, eds. A. Capri and A. Kamal (Plenum Press, 1982); and S. J. Brodsky, T. Huang, and G. P. Lepage, SLAC-PUB-2857, published in the Proceedings of the 1981 SLAC Summer Institute.

5. We are indebted to W. Celmaster and P. Stevenson for emphasizing to us the importance of the scheme choice in our analysis. The general problem of scheme dependence is discussed, for example, in Refs. 1-3. For a recent discussion see A. Blumenfeld and M. Moshe, Phys. Rev. D 26, 648 (1982), where a complete set of references is found.

6. The idea of using a standard theoretical process for defining α_s is not new. For example, the possibility of using deep inelastic lepton moments to define α_s is discussed in A. Para and C. T. Sachrajda, Phys. Lett. 86B, 331 (1979).

7. Vacuum polarization due to fermions with arbitrarily large mass contributes at $Q = 0$ in minimal subtraction schemes. This is compensated by the fact that $\alpha(Q)$ does not tend to the fine structure constant at $Q = 0$ in such schemes. Physically one expects massive fermions to decouple at $Q = 0$ and this can be achieved in minimal subtraction schemes by reorganizing the perturbation theory. The resulting effective coupling constant satisfies Eq. (7). This result is automatic when a conventional subtraction is employed, or when minimal subtraction is used with $n_f = 0$.

8. The Landau singularity, where $\alpha(Q) \to \infty$, does not occur when the UV-divergences are regulated using the Pauli-Villars method, provided the cut-off is finite with $\Lambda \ll e^{3\pi/2\alpha}m$. This is because the running coupling constant stops running for $Q^2 \gtrsim \Lambda^2$.

9. This procedure for QED is equivalent to mass-singularity analyses and renormalization group methods discussed by T. Kinoshita, Nuovo Cimento 51B, 140 (1967); B. E. Lautrup and E. deRafael, Nucl. Phys. B70, 317 (1974); and M. A. Samuel, Phys. Rev. D 9, 2913 (1978). A related method for summing higher-loop QCD contributions to structure function moments in deep inelastic scattering is given in M. Moshe, Phys. Rev. Lett. 43, 1851 (1979) and A. Blumenfeld and M. Moshe (Ref. 5).

10. M. Dine and J. Sapirstein, Phys. Rev. Lett. 43, 668 (1979); K. C. Chetyrkin, A. L. Kataev, F. V. Tkachov, Phys. Lett. 85B, 277 (1979) and Phys. Rev. D 21, 3312 (1980).

11. For a review see A. J. Buras, Rev. Mod. Phys. 52, 1 (1980).

12. S. J. Brodsky and G. P. Lepage, "Quantum Chromodynamics" in the Proceedings of the 1979 Summer Institute on Particle Physics at SLAC, Stanford, California; D. Amati et al., Nucl. Phys. B173, 429 (1980). Some of the ideas used in the present paper are first discussed in the first of these references.

13. R. Barbieri et al., Nucl. Phys. B154, 535 (1979).

14. P. B. Mackenzie and G. P. Lepage, Phys. Rev. 47, 1244 (1981).

15. P. B. Mackenzie and G. P. Lepage, in "Perturbative Quantum Chromodynamics" edited by D. W. Duke and J. F. Owens (AIP Conference Proceedings No. 74, 1981). A thorough analysis of direct photon decays is in preparation.

16. G. P. Lepage and S. J. Brodsky, Phys. Rev. D 22, 2157 (1980) and references therein.

17. R. D. Field et al., Nucl. Phys. B186, 429 (1981); F.-M. Dittes and A. V. Radyushkin, Sov. J. Nucl. Phys. 34, 293 (1981).

18. W. Fischler, Nucl. Phys. B129, 157 (1977); A. Billoire, Phys. Lett. 92B, 343 (1980); W. Buchmuller, G. Grunberg and S.-H. H. Tye, Phys. Rev. Lett. 45, 103 (1980); 45, 587(E) (1980).

19. The MOM scheme based upon the tri-gluon coupling is the most widely used. A different scheme based upon the quark-gluon vertex is adopted by W. Celmaster and R. J. Gonsalves, Phys. Rev. D 21, 3112 (1982).

20. G. Grunberg, Phys. Lett. 95b, 70 (1980) and Ecole Polytechnique Preprint (1982).

21. P. M. Stevenson, Phys. Rev. D 23, 2916 (1981).

Figure Captions

Fig. 1 Diagrams conributing to the muon's anomalous magnetic moment.

Fig. 2 The hard scattering amplitude in leading order for meson form factors.

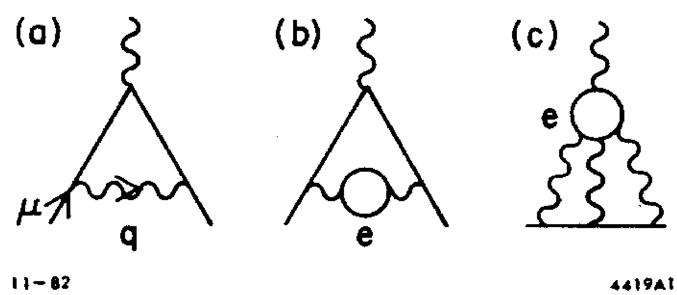

Fig. 1

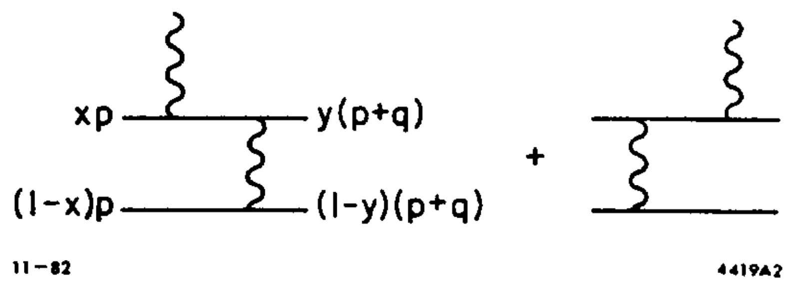

Fig. 2